**Euler on the case $n = 3$ of Fermat's Last Theorem - Continued**

Using our knowledge about Dedekind domains, we prove that, whenever $p$ and $q$ are coprime integers, where $p$ is even and not divisible by $3$, then the number $2p\left(p^2+3q^2\right)$ is never a perfect cube.

Let $\omega = \dfrac{-1+i\sqrt{3}}{2}$ (a primitive cube root of unity). Then $K = \mathbb{Q}(\omega) = \mathbb{Q}\left(i\sqrt{3}\right)$ and its ring of integers is $D_K = \mathbb{Z}\left[\dfrac{1+i\sqrt{3}}{2}\right] = \mathbb{Z}[\omega]$. A basis of $D_K$ over $\mathbb{Z}$ is given by $1,\ \omega$, so that the general form of its elements is $\dfrac{u+iv\sqrt{3}}{2}$, where $u, v$ are integers such that $u \equiv v \pmod{2}$.

**We show that $D_K$ is a PID.** We prove that $cl(D_K) = 1$. To this end, we consider the two embeddings $\sigma_1, \sigma_2 : K \to \mathbb{C}$ (where $\sigma_1$ is the identity) and compute the constant

$$C = \prod_{i=1}^{2} \left(|\sigma_i(1)| + |\sigma_i(\omega)|\right) = \left(1 + \left|\frac{-1+i\sqrt{3}}{2}\right|\right)\left(1 + \left|\frac{-1-i\sqrt{3}}{2}\right|\right) = 2 \cdot 2 = 4.$$

We thus have to consider the prime ideals occurring in the factorizations of the ideals (2) and (3). These factorizations can be determined using Kummer's Criterion. The minimal polynomial of $\omega$ over $\mathbb{Q}$ is $f(x) = \Phi_3(x) = x^2+x+1$, irreducible modulo $2$, whereas it splits as $(x-\overline{1})^2$ modulo $3$. Hence

- (2) is prime in $D_K$,
- $(3) = (3, \omega-1)^2 = (\omega-1)^2$ (because $3 = (\omega-1)(\overline{\omega}-1)$).

All prime ideals we found are principal, which immediately implies our claim.

**Next we show that the numbers $p+q\sqrt{-3}$ and $p-q\sqrt{-3}$ are coprime in $D_K$.** Suppose by contradiction that they have a common prime factor $d$ in $D_K$. Then $d \mid 2p$, but $d \neq 2$, since $d$ divides $p^2+3q^2$, but $2$ does not. Hence $d|p$. This, in turn, implies that $d$ is not a prime factor of $3$: otherwise $p$ and $3$ would not be coprime, against our assumption. On the other hand, $d|3q^2$. Hence $d|q$. But this contradicts the coprimality of $p, q$.

As a consequence of our preceding claim, if the integer

$$p^2+3q^2 = \left(p+qi\sqrt{3}\right)\left(p-qi\sqrt{3}\right)$$

is a a perfect cube, its prime factorization in $D_K$ splits (up to units) into the factorizations of two cubes. Namely, there are two numbers $s = \dfrac{u}{2}$ and $t = \dfrac{v}{2}$ where $u, v$ are integers for which $u \equiv v \pmod{2}$, such that

$$p+qi\sqrt{3} = \left(s+ti\sqrt{3}\right)^3 \qquad (1a)$$

$$p-qi\sqrt{3} = \left(s-ti\sqrt{3}\right)^3 \qquad (1b)$$

(It can be shown that we can neglect units other than $1, -1$.) Note that $(1b)$ follows from $(1a)$ by applying $\sigma_2$. This implies that

$$p^2 + 3q^2 = \left(s^2 + 3t^2\right)^3.$$

Hence, if $2p\left(p^2 + 3q^2\right)$ is a perfect cube, so is $2p$.
From $(1a)$ and $(1b)$ we also deduce that

$$\begin{aligned} 2p &= 2s(s + 3t)(s - 3t) \\ q &= 3t(s + t)(s - t) \end{aligned}$$

The integers $2s$ and $s + 3t$ are coprime. If this were not the case, then $2s$ and $3s + 3t$ would have some common prime factor. This would not be $3$, since $3$ does not divide $p$. But then the same factor would divide $s + t$, so that $p$ and $q$ would not be coprime, against our assumption. It follows that $2s$, $s + 3t$ and $s - 3t$ are coprime. Consequently, if $2p$ is a perfect square, so is each of these factors. But then the conclusion follows as in the previous note.